ISRAEL JOURNAL OF MATHEMATICS, Vol. 48, No. 1, 1984

UPPER BOUNDS FOR PERMANENTS OF (1, −1)-MATRICES

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ABSTRACT

Let Ω_n denote the set of all $n \times n$ $(1, -1)$-matrices. In 1974 E. T. H. Wang posed the following problem: Is there a decent upper bound for $|\operatorname{per} A|$ when $A \in \Omega_n$ is nonsingular? We recently conjectured that the best possible bound is the permanent of the matrix with exactly $n-1$ negative entries in the main diagonal, and affirmed that conjecture by the study of a large class of matrices in Ω_n. Here we prove that this conjecture also holds for another large class of $(1, -1)$-matrices which are all nonsingular. We also give an upper bound for the permanents of a class of matrices in Ω_n which are not all regular.

1. Introduction. Preliminaries

Let Ω_n denote the set of all $n \times n$ $(1, -1)$-matrices and let $\tilde{\Omega}_n \subset \Omega_n$ denote the set of all regular matrices in Ω_n.

Two matrices $A, B \in \Omega_n$ are equivalent, $A \sim B$, if B can be obtained from A by a sequence of the following operations:

(1) interchanging any two rows or columns;

(2) transposition;

(3) negating any row or column.

Clearly, $\sim$ is an equivalence relation and $A \sim B$ implies

$$|\operatorname{per} A| = |\operatorname{per} B|.$$

As Wang showed in [4], the converse is not true.

Obviously, the maximal permanent in Ω_n is $n!$, but the maximal permanent in $\tilde{\Omega}_n$ is unknown in general (E. T. H. Wang asked for it in [4]). In [2] we conjectured an upper bound and proved that our conjecture holds for a large class of matrices. Here we prove the same for another large class of regular matrices (Theorem 1). We also give an upper bound for the permanents of a large class of matrices which are not all regular (Theorem 2). Unfortunately the

Received January 2, 1984

upper bound of Theorem 2 is higher than that of Theorem 1 and hence we could not prove that our conjecture also holds for the regular matrices which are in the class of $(1,-1)$-matrices considered in Theorem 2.

To formulate our results we need the following notations.

Let $C(n,m)=(c_{kj})$, $0\leqq m\leqq n$, denote the $n\times n$ matrix with

$$c_{kj}=\begin{cases} -1 & \text{for } n-m<k=j\leqq n,\\ 1 & \text{otherwise.}\end{cases}$$

$D(n,m)$, $0\leqq m\leqq n$, is the set of those $n\times n$ matrices $A=(a_{kj})$ which are defined by

$$a_{kj}=\begin{cases} -1 & \text{for } n-m<k=j\leqq n,\\ -1 \text{ or } 1 & \text{for } n-m<k<j\leqq n,\\ 1 & \text{otherwise.}\end{cases}$$

Clearly, the $(n-m)$-th row of $A\in D(n,m)$ contains only positive entries. Now, $D'(n,m)$ denotes the set of all $n\times n$ $(1,-1)$-matrices which are defined as the matrices of $D(n,m)$ but with an arbitrary $(n-m)$-th row.

Let A be an $n\times n$ matrix. Then $A(i_1,\ldots,i_r\mid j_1,\ldots,j_s)$ denotes the $(n-r)\times(n-s)$ submatrix which we obtain by deleting the i_1-th, $\ldots$, i_r-th rows and the j_1-th, $\ldots$, j_s-th columns of A.

The following two lemmas were proved in [2].

LEMMA 1.

$$(4)\begin{cases} \operatorname{per} C(n,m)=(n-m-1)\operatorname{per} C(n-1,m-1)+(m-1)\operatorname{per} C(n-1,m-2)\\ \quad \textit{for all } n\geqq 2 \textit{ and for all } 2\leqq m\leqq n,\\ \operatorname{per} C(n,1)=(n-2)(n-1)! \textit{ for all } n\geqq 2;\end{cases}$$

$$(5)\quad\begin{cases} \operatorname{per} C(n,m)=(n-m)\operatorname{per} C(n-1,m)+m\operatorname{per} C(n-1,m-1)\\ \quad \textit{for all } n\geqq 2 \textit{ and for all } 1\leqq m\leqq n-1,\\ \operatorname{per} C(n,0)=n! \textit{ for all } n\geqq 2.\end{cases}$$

LEMMA 2.

$$(6)\qquad \operatorname{per} C(n,m)>0 \qquad \textit{for all } n\geqq 4 \textit{ and for all } 0\leqq m\leqq n.$$

$$(7)\qquad \operatorname{per} C(n,m)<\operatorname{per} C(n,m-1) \qquad \textit{for all } n\geqq 5 \textit{ and for all } 1\leqq m\leqq n.$$

The next lemma is due to H. Perfect [3].

LEMMA 3. *The number of positive products in the permanent expansion*

$$\sum_{\sigma} \prod_{i=1}^{n} a_{i\sigma(i)}$$

of an $n \times n$ $(1,-1)$*-matrix* A *is divisible by* $2^{n-1-[\log_2 n]}$.

We are now able to state and prove our results.

2. Upper bounds

In [2] we conjectured the following upper bounds for the permanents of regular $(1,-1)$-matrices.

CONJECTURE 1. Let $A \in \tilde{\Omega}_n$, $n \geqq 5$. Then

$$|\operatorname{per} A| \leqq \operatorname{per} C(n, n-1) \tag{8}$$

where equality holds iff $A \sim C(n, n-1)$.

If $A \in \tilde{\Omega}_n$, $n \geqq 6$, with $A \not\sim C(n, n-1)$ then

$$|\operatorname{per} A| \leqq \operatorname{per} C(n, n). \tag{9}$$

The following Theorem shows that (8) holds for a large class of regular matrices.

THEOREM 1. *Let* $A \in D(n, m)$, $n \geqq 4$, $1 \leqq m \leqq n-1$. *Then*

$$|\operatorname{per} A| \leqq \operatorname{per} C(n, m). \tag{10}$$

Let $A \in D'(n, m)$, $n \geqq 4$, $1 \leqq m \leqq n-2$. *Then*

$$|\operatorname{per} A| \leqq \operatorname{per} C(n, m). \tag{11}$$

PROOF. For $n = 4$ and $n = 5$ the assertions are checked by detailed discussions of all possible matrices, which can be done within a tolerable amount of time, using a computer.

We now use double induction on n and assume that (10) and (11) hold for all $n \leqq k$, $k \geqq 6$.

Case 1. Let $A \in D(k, m)$ or $A \in D'(k, m)$, $1 \leqq m \leqq k-2$.

$$A = \begin{bmatrix} 1 & 1 & \cdots & & 1 \\ \vdots & & & & \vdots \\ 1 & 1 & \cdots & & 1 \\ \hline & & \pm 1 & & \\ \hline 1 & \cdots & 1 & -1 & \pm 1 \\ \vdots & & & & \\ 1 & 1 & \cdots & 1 & -1 \end{bmatrix}$$

We expand A by the $(k-m)$-th row. Clearly all matrices $A(k-m \mid j) \in D(k-1,m)$ for all $1 \leqq j \leqq k-m$. Now we consider the matrices $A(k-m \mid i)$ for all $k-m+1 \leqq i \leqq k$. We first interchange the $(i-1)$-th row and the $(i-2)$-th row of $A(k-m \mid i)$. Then we interchange the new $(i-2)$-th row with the $(i-3)$-th row, etc., until $i-r=k-m-1$ for some $r \geqq 1$. Hence there is no interchange of rows if $i=k-m+1$ since the $(k-m+1)$-th row of A becomes the $(k-m)$-th row of $A(k-m \mid i)$; there is exactly one interchange of rows if $i=k-m+2$, etc. Now the matrices we obtained from the $A(k-m \mid i)$ by those interchanges are in $D'(k-1,m-1)$. Hence, by induction hypothesis

$$|\operatorname{per} A(k-m \mid j)| \leqq \operatorname{per} C(k-1,m) \quad \text{for all } 1 \leqq j \leqq k-m$$

and

$$|\operatorname{per} A(k-m \mid i)| \leqq \operatorname{per} C(k-1,m-1) \quad \text{for all } k-m+1 \leqq i \leqq k.$$

Now, by expansion and Lemma 1, formula (5), we get

$$|\operatorname{per} A| = \left| \sum_{s=1}^{k} a_{(k-m)s} \operatorname{per} A(k-m \mid s) \right| \leqq \sum_{s=1}^{k} |\operatorname{per} A(k-m \mid s)|$$

$$\leqq (k-m) \operatorname{per} C(k-1,m) + m \operatorname{per} C(k-1,m-1) = \operatorname{per} C(k,m).$$

Case 2. Let $A \in D(k,m)$, $m=k-1$.

Now we expand A by the second row. Clearly, $A(2 \mid 1) = A(2 \mid 2)$ and analogous to Case 1 we can show that all $A(2 \mid i)$, $3 \leqq i \leqq k$, are equivalent to matrices of $D'(k-1,k-3)$. Hence by induction hypothesis and Lemma 1, formula (6), we get

$$|\operatorname{per} A| = \left| \sum_{s=1}^{k} a_{2s} \operatorname{per} A(2 \mid s) \right|$$

$$\leqq \sum_{s=3}^{k} |\operatorname{per} A(2 \mid s)| \leqq (k-2) \operatorname{per} C(k-1,k-3)$$

$$= \operatorname{per} C(k,k-1).$$

In the second sum s only runs from 3 to k because $\operatorname{per} A(2 \mid 1) = \operatorname{per} A(2 \mid 2)$ and $a_{21}=1$, $a_{22}=-1$ hold. □

The next Proposition shows that Theorem 1 affirms parts of Conjecture 1 for a large class of regular $(1,-1)$-matrices.

PROPOSITION 1. *Let $A \in D(n, n-1)$. Then A is regular with*

$$\det A = (-2)^{n-1}. \tag{12}$$

PROOF. If we subtract the first row of A (all entries in this row are equal to $+1$) from all other rows we get the matrix $X = (x_{ij})$, where

$$x_{ij} = \begin{cases} 0, & \text{for } 1 \leqq j < i \leqq n, \\ -2 & \text{for } 2 \leqq i = j \leqq n, \\ 1 & \text{for } i = j = 1, \end{cases}$$

for which $\det X = \det A = (-2)^{n-1}$ clearly holds. □

Unfortunately (10) and (11) do not hold for $m = n$ and $m = n-1$, respectively. For example, the following matrix $A \in D(n, n)$ is equivalent to $C(n, n-2)$ and thus has a permanent greater than per $C(n, n-1)$ by Lemma 2, inequality (7):

$$A = \begin{bmatrix} -1 & -1 & -1 & \cdots & \cdots & -1 & 1 \\ 1 & -1 & 1 & \cdots & \cdots & 1 & -1 \\ 1 & 1 & -1 & 1 & \cdots & 1 & -1 \\ \vdots & \vdots & & & & \vdots & \vdots \\ 1 & 1 & \cdots & \cdots & \cdots & -1 & -1 \\ 1 & 1 & & & & 1 & -1 \end{bmatrix}$$

This, and a detailed discussion of all matrices of Ω_5, immediately leads to the conjecture that per $C(n, n-2)$ is the best upper bound for the matrices of $D'(n, n-1)$ $(D(n, n) \subset D'(n, n-1))$. As the next theorem shows, this conjecture holds.

THEOREM 2. *Let $A \in D'(n, n-1)$, $n \geqq 4$. Then*

$$|\operatorname{per} A| \leqq \operatorname{per} C(n, n-2). \tag{13}$$

For $n \geqq 6$ equality cannot hold if at least one of the following conditions is satisfied:

(i) $a_{11} \neq a_{12}$;

(ii) $\operatorname{sgn}(\operatorname{per} A(1|1)) \neq \operatorname{sgn}(\operatorname{per} A(1|2))$;

(iii) $\operatorname{per} A(1|i) = 0$ *for at least one* $i \in \{1, 2\}$.

PROOF. For $n = 4$ and $n = 5$ the result is again proved by a detailed discussion of all possible matrices. We now assume that (13) holds for all $n < k$, $k \geqq 6$, and use induction on n.

Let $A \in D'(k, k-1)$.

$$A = \begin{bmatrix} \boxed{\qquad\qquad\quad \pm 1 \qquad\qquad\quad} \\ 1 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 & -1 & \pm 1 \\ \vdots \\ 1 & 1 & 1 & \cdots & 1 & -1 \end{bmatrix}$$

We expand A by the first row and analogous to Theorem 1, Case 1, we can show that the $A(1\,|\,j)$, $1 \leqq j \leqq k$, are equivalent to matrices of $D'(k-1, k-2)$.

Hence by induction hypothesis

$$(14) \qquad |\operatorname{per} A(1\,|\,j)| \leqq \operatorname{per} C(k-1, k-3) \qquad \text{for all } j.$$

Since

$$\operatorname{per} C(k, k-2) = (k-2)\operatorname{per} C(k-1, k-3) + 2\operatorname{per} C(k-1, k-2)$$

we need more than inequality (14) to prove our result. Therefore we consider $X = A(1\,|\,1)$ and $Y = A(1\,|\,2)$ more precisely.

Case 1. $\operatorname{sgn}(\operatorname{per} X) \neq \operatorname{sgn}(\operatorname{per} Y)$.

Only $x_{11} \neq y_{11}$, otherwise X and Y have the same entries.

So $X(1\,|\,1) = Y(1\,|\,1)$ and obviously

$$X(1\,|\,1) = Y(1\,|\,1) \in D'(k-2, k-3)$$

also holds. Hence

$$|\operatorname{per} X(1\,|\,1)| = |\operatorname{per} Y(1\,|\,1)| \leqq \operatorname{per} C(k-2, k-4)$$

by induction hypothesis. Since $\operatorname{sgn}(\operatorname{per} X) \neq \operatorname{sgn}(\operatorname{per} Y)$ this implies

$$|\operatorname{per} X| + |\operatorname{per} Y| \leqq 2\operatorname{per} C(k-2, k-4).$$

As, by Lemma 1, formula (4),

$$\operatorname{per} C(k-1, k-2) = (k-3)\operatorname{per} C(k-2, k-4)$$

the inequality

$$(15) \qquad 2\operatorname{per} C(k-2, k-4) < \operatorname{per} C(k-1, k-2)$$

holds for all $k \leqq 6$. By (14) we get

$$\sum_{s=3}^{k} |\operatorname{per} A(1|s)| \leqq (k-2)\operatorname{per} C(k-1, k-3)$$

and hence together with (5)

$$|\operatorname{per} A| < \operatorname{per} C(k-1, k-2) + (k-2)\operatorname{per} C(k-1, k-3) < \operatorname{per} C(k, k-2).$$

Case 2. If either $\operatorname{per} X = 0$ or $\operatorname{per} Y = 0$ we get the inequality of Case 1; if $\operatorname{per} X = \operatorname{per} Y = 0$ then our result obviously holds.

Case 3. $\operatorname{sgn}(\operatorname{per} X) = \operatorname{sgn}(\operatorname{per} Y)$.

We again expand X and Y by the first row.

$$X = \begin{bmatrix} -1 & & & & & \\ 1 & -1 & & & & \\ 1 & 1 & -1 & \pm 1 & & \\ \vdots & & & & & \\ 1 & 1 & \cdots & & 1 & -1 \end{bmatrix}, \qquad Y = \begin{bmatrix} 1 & & & & & \\ 1 & -1 & & & & \\ 1 & 1 & -1 & \pm 1 & & \\ \vdots & & & & & \\ 1 & 1 & \cdots & & 1 & -1 \end{bmatrix}$$

Obviously, the submatrices $X(1|j) = Y(1|j)$ are equivalent to matrices of $D'(k-2, k-3)$ for all $1 \leqq j \leqq k-2$ and $X(1|k-1) = Y(1|k-1)$ are equivalent to a matrix in $D(k-2, k-3)$.

Now, if $a_{11} \neq a_{12}$ then

$$|a_{11} \operatorname{per} X + a_{12} \operatorname{per} Y| = 2|\operatorname{per} X(1|1)| \leqq 2\operatorname{per} C(k-2, k-4)$$

holds since X and Y have equal entries with the exception $x_{11} \neq y_{11}$. Hence, as in Case 1, we get

$$|\operatorname{per} A| < \operatorname{per} C(k, k-2).$$

If $a_{11} = a_{12}$ then

$$\begin{aligned} |a_{11} \operatorname{per} X + a_{12} \operatorname{per} Y| &= \left| 2 \sum_{s=2}^{k-1} x_{1s} \operatorname{per} X(1|s) \right| \leqq 2 \sum_{s=2}^{k-1} |\operatorname{per} X(1|s)| \\ &\leqq 2(k-3)\operatorname{per} C(k-2, k-4) + 2\operatorname{per} C(k-2, k-3) \\ &= \operatorname{per} C(k-1, k-2) + \operatorname{per} C(k-1, k-3) \end{aligned}$$

since $X(1|1) = Y(1|1)$, $x_{11} \neq y_{11}$ and $X(1|k-1) = Y(1|k-1)$ are equivalent to a matrix of $D(k-2, k-3)$.

Obviously, $A(1|k)$ is equivalent to a matrix of $D(k-1, k-2)$ and hence

$$|\operatorname{per} A(1\,|\,k)| \leqq \operatorname{per} C(k-1, k-2)$$

by Theorem 1.

So

$$\begin{aligned}
|\operatorname{per} A| &= \left| \sum_{s=1}^{k} a_{1s} \operatorname{per} A(1\,|\,s) \right| \\
&\leqq |a_{11} \operatorname{per} A(1\,|\,1) + a_{12} \operatorname{per} A(1\,|\,2)| + \sum_{s=3}^{k} |\operatorname{per} A(1\,|\,s)| \\
&\leqq 2\operatorname{per} C(k-1, k-2) + (k-2)\operatorname{per} C(k-1, k-3) = \operatorname{per} C(k, k-2). \quad \square
\end{aligned}$$

The next lemma, due to H. Perfect [3], gives an explicit formula for our upper bounds.

LEMMA 4.

$$\operatorname{per} C(n, m) = n! - 2 \sum_{k=1}^{m} (-1)^{k-1} 2^{k-1} \frac{(n-k)!\,m!}{(m-k)!\,k!}$$

for $n \geqq 2$ *and all* $0 \leqq m \leqq n$.

In Proposition 1 we proved that all matrices of $D(n, n-1)$ are regular. Now we can ask for the converse: How many regular $n \times n$ $(1, -1)$-matrices are equivalent to a matrix of $D(n, n-1)$ or $D'(n, n-1)$ and thus have permanents bounded by Theorem 1 or Theorem 2?

This question can be answered for $n = 3, 4$ and 5.

PROPOSITION 2. *All matrices of* $\tilde{\Omega}_n$ *are equivalent to matrices of* $D(n, n-1)$ *or* $D'(n, n-1)$ *for all* n, $3 \leqq n \leqq 5$.

PROOF. A detailed discussion of all matrices of Ω_3, Ω_4 and Ω_5 leads to the sets $\tilde{\Omega}_3$, $\tilde{\Omega}_4$ and $\tilde{\Omega}_5$ as they are listed below (the sets $\tilde{\Omega}_3$, $\tilde{\Omega}_4$ and $\tilde{\Omega}_5$ are clearly presented modulo $\sim$).

$$\tilde{\Omega}_3: \quad \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$\tilde{\Omega}_4: \quad \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix} \qquad \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}$$

$$\tilde{\Omega}_5:\quad \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} -1 & -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 \end{bmatrix} \qquad \square$$

We could not prove a result like Proposition 3 for $n \geqq 6$, but we think that it also holds for larger n.

CONJECTURE 2. All matrices of $\tilde{\Omega}_n$ are equivalent to matrices of $D(n, n-1)$ or $D'(n, n-1)$ for all $n \geqq 3$.

At last we present a Proposition which shows that for $n = 6$ the second part of Conjecture 1 immediately follows if the first part holds.

PROPOSITION 3. *Let* $A \in \Omega_6$. *If*

$$|\operatorname{per} A| < \operatorname{per} C(6,5)$$

then

$$|\operatorname{per} A| \leqq \operatorname{per} C(6,6)$$

also holds.

PROOF. By the formulae of Lemma 1 or Lemma 4 it is easy to derive that

$$\operatorname{per} C(6,5) = 128$$

and

$$\operatorname{per} C(6,6) = 112.$$

Now suppose

$$|\operatorname{per} A| < 128.$$

As we have shown in [1] per B is divisible by $2^{n-1-[\log_2 n]}$ for all $B \in \Omega_n$.

So, if our assertion does not hold, then $|\operatorname{per} A| = 120$. But $|\operatorname{per} A| = 120$ implies that the number of positive products in the permanent expansion of A is equal to 420 or 300. In both cases the number of positive products is not divisible by 8, a contradiction to Lemma 3. Hence $|\operatorname{per} A| \leqq 112 = \operatorname{per} C(6,6)$. □

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